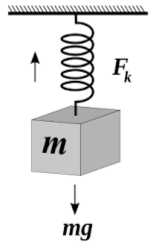
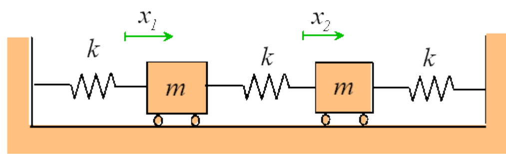
Lecture 11: Case Studies—Linear Algebraic Equations



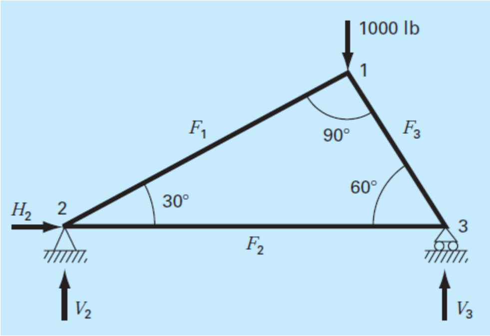


Cases

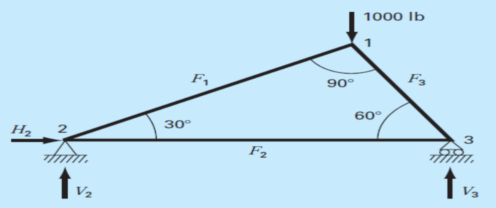
1. Analysis of Truss
2. Spring-Mass System

Analysis of a Statically Determinate Truss

- **Background**. An important problem in structural engineering is that of finding the forces and reactions associated with a statically determinate truss.



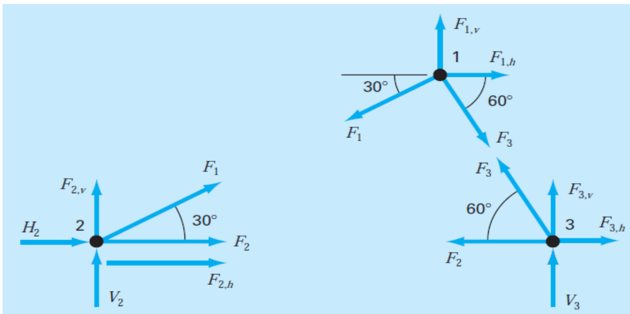
Analysis of a Statically Determinate Truss



- The forces (F) represent either tension or compression on the members of the truss.
- External reactions (H_2, V_2, and V_3) are forces that characterize how the truss interacts with the supporting surface.
- The hinge at node 2 can transmit both horizontal and vertical forces to the surface, whereas the roller at node 3 transmits only vertical forces.
- It is observed that the effect of the external loading of 1000 lb is distributed among the various members of the truss.

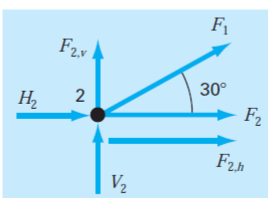
Analysis of a Statically Determinate Truss

- **Solution**. This type of structure can be described as a system of coupled linear algebraic equations.
- Free-body force diagrams are shown for each node in the figure below.



Analysis of a Statically Determinate Truss

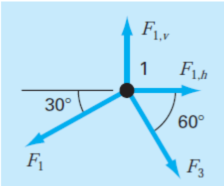
- The sum of the forces in both horizontal and vertical directions must be zero at each node, because the system is at rest.
- Therefore, for node 1,



$$\Sigma F_H = 0 = -F_1 \cos 30^\circ + F_3 \cos 60^\circ + F_{1,h}$$
$$\Sigma F_V = 0 = -F_1 \sin 30^\circ - F_3 \sin 60^\circ + F_{1,v}$$

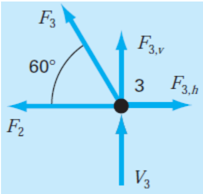
Analysis of a Statically Determinate Truss

- For node 2,



$$\Sigma F_H = 0 = F_2 + F_1 \cos 30^\circ + F_{2,h} + H_2$$
$$\Sigma F_V = 0 = F_1 \sin 30^\circ + F_{2,v} + V_2$$

- For node 3,



$$\Sigma F_H = 0 = -F_2 - F_3 \cos 60^\circ + F_{3,h}$$
$$\Sigma F_V = 0 = F_3 \sin 60^\circ + F_{3,v} + V_3$$

Analysis of a Statically Determinate Truss

- where $F_{i,h}$ is the external horizontal force applied to node *i* (where a positive force is from left to right)
- and $F_{i,v}$ is the external vertical force applied to node *i* (where a positive force is upward).
- Note that the directions of the internal forces and reactions are unknown.
- Also note that in this problem, the forces in all members are assumed to be in tension and act to pull adjoining nodes together.
- A negative solution therefore corresponds to compression.

Analysis of a Statically Determinate Truss

- This problem can be written as the following system of six equations and six unknowns:

$$\begin{bmatrix} 0.866 & 0 & -0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.866 & 0 & 0 & 0 \\ -0.866 & -1 & 0 & -1 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.866 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ H_2 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$F_1 = -500 \quad F_2 = 433 \quad F_3 = -866$$
$$H_2 = 0 \quad V_2 = 250 \quad V_3 = 750$$

Which is your preferred method if you need to solve the system for varied externally applied horizontal and vertical forces?

Spring-Mass System

- When a spring is stretched or compressed by a mass, the spring develops a restoring force.
- *Hooke's law* gives the relationship of the force exerted by the spring when the spring is compressed or stretched a certain length:

$$F(t) = -kx(t)$$

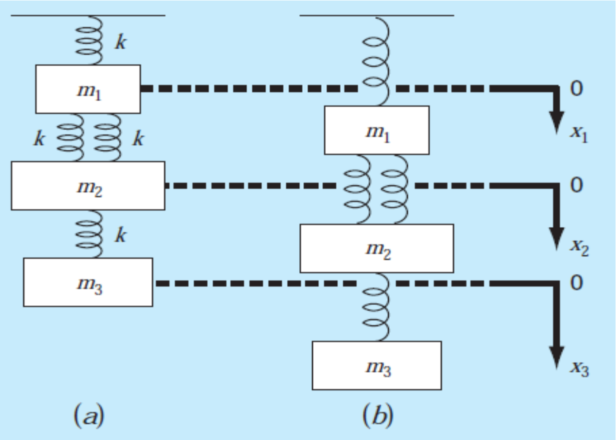
- where *F* is the force, *k* is the spring constant, and *x* is the displacement of the mass with respect to the equilibrium position.

Spring-Mass System

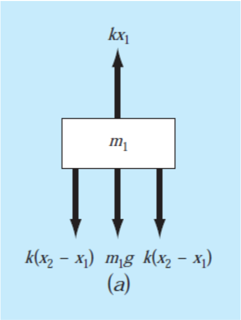
- By using force balance, it can be readily shown that the motion of this system is given by the following differential equation:

$$m\frac{d^2x}{dt^2} = F_D - F_U$$

Spring-Mass System



FBD for the first mass

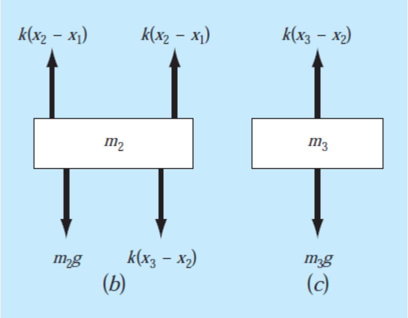


$$F_U = kx_1$$

$$F_D = k(x_2 - x_1) + k(x_2 - x_1) + m_1 g$$

$$m_1 \frac{d^2 x_1}{dt^2} = 2k(x_2 - x_1) + m_1 g - kx_1$$

FBD for the last two masses



$$m_2 \frac{d^2 x_2}{dt^2} = k(x_3 - x_2) + m_2 g - 2k(x_2 - x_1)$$

$$m_3 \frac{d^2 x_3}{dt^2} = m_3 g - k(x_3 - x_2)$$

Solutions when system comes to rest

$$\begin{aligned} 3kx_1 - 2kx_2 \quad\quad &= m_1 g \\ -2kx_1 + 3kx_2 - kx_3 &= m_2 g \\ - kx_2 + kx_3 &= m_3 g \end{aligned}$$

$$[K]\{X\} = \{W\}$$

$$[K] = \begin{bmatrix} 3k & -2k & \\ -2k & 3k & -k \\ & -k & k \end{bmatrix}$$

Solutions when system comes to rest

$$\begin{aligned} 3kx_1 - 2kx_2 \quad\quad &= m_1 g \\ -2kx_1 + 3kx_2 - kx_3 &= m_2 g \\ - kx_2 + kx_3 &= m_3 g \end{aligned}$$

If m_1 = 2 kg, m_2 = 3 kg, m_3 = 2.5 kg, and the k's = 10 kg/s^2

$$x_1 = 7.35, x_2 = 10.045, \text{and } x_3 = 12.495.$$

$$[K]^{-1} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.15 & 0.15 \\ 0.1 & 0.15 & 0.25 \end{bmatrix}$$

What does an element of the inverse matrix mean?

Assignment-11

- Problems 12.16, 12.33, 12.34.